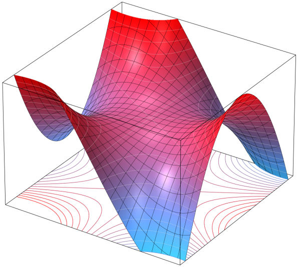
Perturbing the monkey's saddle

The **monkey saddle** $\Re(z^3) = x^3 - 3xy^2$ is not Morse:



- How many different natural and interesting ways can you think of to perturb it to be Morse?
- How are these related by generic homotopies of functions? What kinds of singularities do you encounter?
- How are these homotopies related by generic homotopies of homotopies? What kinds of singularities appear now?

Indefinite Morse 2–functions

(and some reflections on the monkey saddle)

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Morse functions and Cerf theory

- **Morse function:** Generic $g\colon M^m \to Y^1$, $g(\partial M) \subset \partial Y$
  - Usual local models
  - Isolated critical points, isolated critical values
- **Cerf theory:** Generic homotopy $g_t\colon M^m \to Y^1$ between Morse functions g_0, g_1 is Morse except at discrete values $t = t_*$
  - Local models when g_{t_*} not Morse: critical values cross, births/deaths
  - **Cerf graphic:** (visualization tool) image of 1–dimensional singular locus of $(t,p) \mapsto (t, g_t(p))$, $I \times M \to I \times Y$
- **Two parameters:** Generic homotopy $g_{s,t}$ between generic homotopies $g_{0,t}$ and $g_{1,t}$ between Morse functions is a generic homotopy between Morse functions except at discrete values $s = s_*$
  - Local models when $g_{s_*,t}$ not generic: well understood, more later
- **Indefinite:** no minimal or maximal index critical points

Morse 2–functions

- **Morse 2–function:** Generic map $G\colon X^n \to \Sigma^2$, $G(\partial X) \subset \partial\Sigma$
  - Example: $X = I \times M^{n-1}$, $\Sigma = I \times Y^1$, $G(t,p) = (t, g_t(p))$ for generic $g_t\colon M^{n-1} \to Y^1$
  - Locally all Morse 2–functions look like this example
- **"Cerf theory" for Morse 2–functions:** Study generic homotopies $G_s\colon X^n \to \Sigma^2$ between Morse 2–functions G_0 and G_1
  - Example: $X = I \times M^{n-1}$, $\Sigma = I \times Y^1$, $G_s(t,p) = (t, g_{s,t}(p))$ for generic homotopy of homotopies $g_{s,t}\colon M^{n-1} \to Y^1$
  - Locally all generic homotopies between Morse 2–functions look like this example
  - In particular, G_s is a Morse 2–function except at discrete values $s = s_*$
- **Indefinite:** no minimal or maximal index critical points in local models

Indefinite S^2–valued Morse 2–functions

Theorem

- ***Existence*** *($n \geq 3$): Given*
  - *Closed (connected) X^n*
  - *Framed (connected) $F^{n-2} \subset X$*

  Then
  - $\exists$ indefinite *Morse* 2*–function $G\colon\ X \to S^2$ with $G^{-1}(\mathrm{n.p.}) = F^{n-2}$ (framed) (*Saeki*)*
  - *(All fibers of G are connected)*
- ***Uniqueness*** *($n \geq 4$): Given*
  - *$G_0, G_1\colon\ X \to S^2$ homotopic, indefinite Morse 2–functions*
  - *(All fibers of G_0 and G_1 connected)*

  Then:
  - $\exists$ indefinite *generic homotopy $G_s\colon\ X \to S^2$ between G_0 and G_1 (*Williams*)*
  - *(All fibers of G_s are connected for all s)*

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  - *(All fibers of G are connected)*
- ***Uniqueness*** *($n \geq 4$): Given*
  - $G_0, G_1\colon\ X \to S^2$ *homotopic, indefinite Morse* 2*–functions*
  - *(All fibers of G_0 and G_1 connected)*

  Then:
  - $\exists$ indefinite *generic homotopy* $G_s\colon\ X \to S^2$ *between G_0 and G_1 (*Williams*)*
  - *(All fibers of G_s are connected for all s)*

Indefinite D^2–valued Morse 2–functions

Theorem

- ***Existence*** *($n \geq 3$): Given*
  - *Compact, connected X^n, $\partial X \neq \emptyset$*
  - *Indefinite Morse $g\colon \partial X \to S^1$ (with connected level sets)*

  Then
  - $\exists$ indefinite *Morse 2–function $G\colon X \to D^2$ with $G|_{\partial X} = g$*
  - *(All fibers of G are connected)*
- ***Uniqueness*** *($n \geq 4$): Given*
  - *X, g as above*
  - *$G_0, G_1\colon X \to D^2$ indefinite Morse 2–functions with $G_0|_{\partial X} = G_1|_{\partial X} = g$*
  - *(All fibers of G_0 and G_1 connected)*

  Then:
  - $\exists$ indefinite *generic homotopy $G_s\colon X \to D^2$ between G_0 and G_1, with $G_s|_{\partial X} = g$ for all s*
  - *(All fibers of G_s are connected for all s)*

Indefinite D^2–valued Morse 2–functions

Theorem

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  - *Compact, connected X^n, $\partial X \neq \emptyset$*
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  - *(All fibers of G_s are connected for all s)*

Indefinite I^2–valued Morse 2–functions

Coordinates on $I^2 = I \times I$: (t, z)

Theorem

- ***Existence*** *($n \geq 3$): Given*
  - *Closed* F_{ij}^{n-2} *for* $i, j \in \{0, 1\}$, $F_{0j} \cong F_{1j}$
  - *Cobordisms* M_i^{n-1} *from* F_{i0} *to* F_{i1}, $i = 0, 1$
  - *Indefinite Morse* $g_i : M_i \to I$, $i = 0, 1$, *(with connected level sets)*
  - *Connected cobordism-with-sides* X^n *from* M_0 *to* M_1, *i.e.* $\partial X_n = -M_0 \cup (I \times F_{00}) \cup (I \times (-F_{01})) \cup M_1$

Then

  - $\exists$ indefinite *Morse 2–function* $G : X \to I^2$ *such that:*
  - $G(M_i) = \{i\} \times I$
  - $z \circ G|_{M_i} = g_i$
  - $t \circ G|_{I \times (\pm F_{0i})}$ *is projection* $I \times (\pm F_{0i}) \to I$
  - *(All fibers of G are connected)*

Indefinite I^2–valued Morse 2–functions continued

Theorem

- ***Uniqueness*** $(n \geq 4)$: *Given*
  - X, M_i, F_{ij}, g_i *as above*
  - $G_0, G_1 : X \to I^2$ *satisfying conditions satisfied by G above*
  - *(All fibers of* G_0 *and* G_1 *connected)*

  Then:
  - $\exists$ indefinite *generic homotopy* $G_s : X \to I^2$ *from* G_0 *to* G_1*, with* $G_s|_{\partial X} = G_0|_{\partial X}$ *for all s*
  - *(All fibers of* G_s *connected for all s)*

- **Easy:** I^2–valued existence $\implies$ D^2–valued existence $\implies$ S^2–valued existence.
- **Easy:** I^2–valued uniqueness $\implies$ D^2–valued uniqueness.
- **Not totally trivial:** D^2–valued uniqueness $\implies$ S^2–valued uniqueness.

Morse and Cerf theory ingredients:

Theorem

Given connected cobordism M^m from closed, nonempty (connected) F_0^{m-1} to F_1^{m-1}, with framed submanifold $K^k \subset M$, $k \leq m-2$:

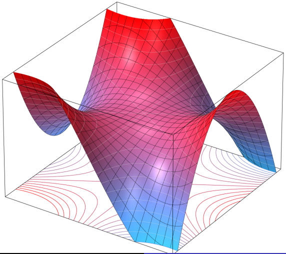
- ***Existence:*** *$(m \geq 1)$* $\exists$ indefinite *Morse function $g\colon M \to I$ with $g(F_j) = j$, with K framed in level set(s).*
- ***Uniqueness:*** *$(m \geq 2)$ Given indefinite Morse functions $g_0, g_1\colon M \to I$ with $g_i(F_j) = j$, with K framed in level set(s),* $\exists$ indefinite *generic homotopy $g_t\colon M \to I$ between g_0 and g_1* (Kirby) *keeping K framed in the level set(s).*
- ***Two-parameter uniqueness:*** *$(m \geq 3)$ Given g_0, g_1 as above, and two indefinite generic homotopies $g_{0,t}, g_{1,t}\colon M \to I$ between $g_0 = g_{0,0} = g_{1,0}$ and $g_1 = g_{0,1} = g_{1,1}$,* $\exists$ indefinite *generic homotopy $g_{s,t}\colon M \to I$ between $g_{0,t}$ and $g_{1,t}$, with $g_{s,0} = g_0$ and $g_{s,1} = g_1$ for all s.*

Ideas of proofs

Proof.

- **Existence:** Get K in level set(s), cancel 0–handles avoiding K
- **Uniqueness:** Cancel 0–handles and swallowtails avoiding K
- **Two-parameter uniqueness:** Cancel 0–handles and swallowtails, then *flip monkey saddles*.

□



Morse 2–function building blocks:

Existence of indefinite I^2–valued Morse 2–functions on X^n follows from:

- Existence for indefinite Morse functions on X^n
- Uniqueness for indefinite Morse functions on M^{n-1}
- The following theorem:

Theorem

Given F_{ij}^{n-2}, M_i^{n-1}, X^n, with:

- *Morse function $\tau\colon X \to I$ with exactly one critical point of index $\leq n-2$, with framed attaching sphere $K \subset M_0$*
- *Morse function $g\colon M_0 \to I$ with K framed in a level set*

Then $\exists$ Morse 2–function $G\colon X \to I$ with:

- $t \circ G = \tau$
- $z \circ G|_{M_0} = g_0$

Ingredients for indefinite Morse 2–function uniqueness:

Given F_{ij}, M_i, X, indefinite $G_0, G_1 : X \to I^2$ which agree on M_i

Theorem

If $t \circ G_0 = t \circ G_1$ *then* $\exists$ indefinite *generic homotopy* G_s *between* G_0 *and* G_1 *with* $t \circ G_s = t \circ G_0$ *for all* s.

Theorem

Let $\tau_0 = t \circ G_0, \tau_1 = t \circ G_1 : X \to I$*. Given an indefinite generic homotopy* τ_s *between* τ_0 *and* τ_1*,* $\exists$ *an indefinite generic homotopy* G'_s *between* $G'_0 = G_0$ *and* G'_1 *with* $t \circ G'_s = \tau_s$*.*